MODULE - I

Sets of objects, numbers, departments, job descriptions, etc. are things that we all deal with every day of our lives. Mathematical Set Theory just puts a structure around this concept so that sets can be used or manipulated in a logical way. The type of notation used is a reasonable and simple one.

For example, suppose a company manufactured 5 different products a, b, c, d, and e. Mathematically, we might identify the whole set of products as *P*, say, and write:

$P = (a,b,c,d,e)$

which is translated as 'the set of company products, P, consists of the *members (or elements*) *a, b, c, d* and *e.*

The elements of a set are usually put within braces (curly brackets) and the elements separated by commas, as shown for set P above.

A mathematical set is a collection of distinct objects, normally referred to as *elements* or *members.*

The theory of sets was introduced by the German mathematician Georg Cantor in 1870. That is, there exists a rule with the help of which we will be able to say whether a particular object 'belong to' the set or does not belong to the set..

The sets are usually denoted by the Capital letters of the English alphabet and the elements are denoted by small letters. The objects in a set are called its members or elements of the sets.

Eg. for sets: Vowels in alphabets, students in class, flowers in garden

Representation of a Set or Methods of describing a Set

A set is often representation in two ways:

(1) Roster method or tabular or enumeration method.

(2) Set builder method or Rule or Selector method.

Tabular Method

In this method, a set is described by listing the elements, separated by commas and are enclosed within braces. For example the set of first three odd numbers 1,3,5 is represented as :

A = $\{1, 3, 5\}$

Set Builder Method

In this method, the set is represented by specifying the characteristic property of its elements. For example the set of natural numbers between 1 and 25 is represented as:

A = $\{x: x \in N \text{ } and \text{ } 1 < x < 25\}$

TYPES OF SETS

1. Null Set or Empty Set or Void Set

A set containing no element is called a null set. It is denoted by { } or Ø

Eg:- the set of natural numbers between 4 and 5.

2. Singleton or Unit Set

A Set containing a single element is called singleton set

Eg:- Set of all positive integers less than 2

3. Finite Set

A Set is said to be a finite set if it consist only a finite number of elements. The null set is regarded as a finite set.

Eg:- the set of natural numbers less than 10

4. Infinite Set

A set is said to be an infinite set if it consists of a infinite number of elements.

Eg:- Set of natural numbers.

5. Equvilant Set

Two sets A and B are said to be equivalent set if they contain the same number of elements

Eg:- Let A = {1, 2, 3 } and B = $\{a, b, c\}$

6. Equal Set

Two sets A and B are said to be equal if they contain the same elements.

Eg:- Let A = {1, 2, 3 } B = {2, 1, 3 }

7. Sub Set and Super Set

If every element of A is an element of B then A is called a subset of B and symbolically we write $A \subseteq B$

If A is contain in B then B is called super set of A and written as $B \supseteq A$

Eg: A = {2, 3 } and B = {2, 3, 4 } then A is a proper subset of B

8. Power Set :-

The collection of all sub sets of a set A is called the power set of A. It is denoted by P(A). In P(A), every element is a set. For example

A = {1, 2, 3 }

Then P(A) ={ },{1, 2, 3 } {1}{ 2, } {3 }{1, 2 } {1, 3 } {2, 3 }

9. Universal Set

If all the sets under consideration are subsets of a fixed set U, is called universal set. For example A is the set of vowels in the English Alphabet. Then the set of all letters of the English Alphabet may be taken as the universal set.

10. Disjoint Set

Two sets A and B are said to be disjoint sets if no element of A is in B and no element of B is in A. For example

A= {3, 4, 5 }, B = {6, 7, 8 }

SET OPERATIONS

(1) Union of sets :
The union of two sets A and B is the set of all those elements which belongs to A or to B or to both. We use the notation $A \cup B$ to denote the union of A and B.

For example

If A = {1, 2, 3, 4 } B = {3, 4, 5, 6 } , Then A∪B = {1, 2, 3, 4, 5, 6 }

(2) Intersection of Sets

The intersection of two sets is the set consisting of all elements which belong to both A and B. It is denoted by A∩B. For example

If A = {1, 2, 3, 4 } B = {3, 4, 5, 6 } , Then A∩B = {3, 4 }

(3) Difference of two sets

The difference of the two sets A and B is the set of all elements in A which are not in B. It is denoted by A-B or A/B. For example

If A = {1, 2, 3, 4 } B = {3, 4, 5, 6 } , Then A—B = {1, 2 }

(4) Complement of a set

Complement of a set is the set of all element belonging to the universal set but not belonging to A. It is denoted by A^c or A'

A^c = ∪-A. For example If ∪= {1, 2, 3, 4,5 } A = {1,3, 5} , Then A^c = { 2, 4 }

ALGEBRA OF SETS OR LAWS OF SET OPERATION

(1) Commutative Laws :-

If A and B are any two sets then :-

(i)A∪B =B∪A

(ii)A∩B = B∩A

(2) Associative Laws

If A, B and C are three sets, then

(i)A∪ (B∪C) = (A∪B) ∪C and

(ii)A∩ (B∩C) = (A∩B) ∩C

Distributive Laws

If A, B, C are any three sets, then

(i) A∪ (B∩C) = (A∪B) ∩(A∪C) and

(ii) A∩ (B∪C) = (A∩B) ∪ (A ∩C)

De-Morgan's Law

If A and B are any two subsets of '∪', then

(i)(A∪B)' =A'∩B'

That is complement of union of two sets equal to the intersection of their complements.

(ii) $(A \cap B)' = A' \cup B'$

That is complement of intersection of two sets is equal to the union of their complements.

Practical Problems

1) If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$

$C = \{5, 6, 7, 8\}$ $D = \{7, 8, 9, 10\}$

Find (i) $A \cup B$ (ii) $A \cup C$ (iii) $B \cup C$

(iv) $B \cup D$ (v) $A \cup B \cup C$ (vi) $A \cup B \cup D$ (vii) $B \cup C \cup D$

Answer

(i) $A \cup B = \{1, 2, 3, 4, 5, 6\}$

(ii) $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\}$

(iii) $B \cup C = \{3, 4, 5, 6, 7, 8\}$

(iv) $B \cup D = \{3, 4, 5, 6, 7, 8, 9, 10\}$

(v) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\}$

(vi) $A \cup B \cup D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(vii) $B \cup C \cup D = \{3, 4, 5, 6, 7, 8, 9, 10\}$

2) If $A = \{1, 3, 5, 7\}$, $B = \{5, 9, 13, 17\}$ $C = \{1, 3, 9, 13\}$

Find (i) $A \cap B$ (ii) $B \cap A$ (iii) A-B (iv) $B - A$ (v) $A - C$ (vi) $(A-B) - C$ (vii) $A-(A-B)$

Answer

(i) $A \cap B = \{5\}$

(ii) $B \cap A = \{5\}$

(iii) $A\text{-}B = \{1, 3, 7\}$

(iv) $B\text{-}A = \{9, 13, 17\}$

(v) $A - C = \{5, 7\}$

(vi) $(A-B) - C) = \{7\}$

(vii) $A\text{-}(A\text{-}B) = \{5\}$

3) $A = \{x : x \text{ is a natural number satisfy } 1 < x \leq 6\}$

$B = \{x : x \text{ is a natural number satisfy } 6 < x \leq 10\}$

Find (i) $A \cup B$ (ii) $A \cap B$

Answer

$A = \{2, 3, 4, 5, 6\}$

B = {7,8, 9, 10 }

(i) A∪B = {2, 3, 4, 5, 6, 7,8, 9, 10 }

(ii) A∩B = { }

4) Let ∪ = { 1,2, 3, 4, 5, 6, 7, 8, 9,10 } and A = {1, 3, 5, 7, 9 } Find A^C.

Answer

A^C means belongs to universe but not in A

A^C = {2, 4, 6, 8, 10}

5) Let ∪= { 1,2, 3, 4, 5, 6, 7, 8, 9,10 }

A = { 1, 4, 7,10 } B = { 2, 4, 5, 8 }

Find A'∩B

Answer:

A' = Belongs to universe but not in A

A' = { 2, 3, 5, 6, 8, 9 }

A'∩B = { 2, 5, 8 }

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7) Let A = { 1,2,3 } B = { 2, 4, 5}

C = { 2, 4, 6}, ∪= { 1, 2, 3, 4, 5, 6, 7, 8 }

Verify that (i) (A∩B) = A'∩B'

(ii) (A∩B) = A'∪B'

Answer

(i) (A∪B)' = ∪ − (A∪B)

A∪B = { 1, 2, 3, 4, 5}

(A∪B)' = { 6, 7, 8}

A' = ∪ −A = { 4, 5, 6, 7, 8}

B' = ∪ −B = { 1, 3, 6, 7, 8}

A'∩B' = { 6, 7, 8}

Hence (A∪B)' = A'∩B'

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(ii) (A∩B)' = ∪ − (A∩B)

(A∩B) = { 2}

(A∩B)' = { 1, 3, 5, 6, 7, 8}

A'∪B' = { 1, 3, 5, 6, 7, 8}

Hence (A∩B)'= A'∪B'

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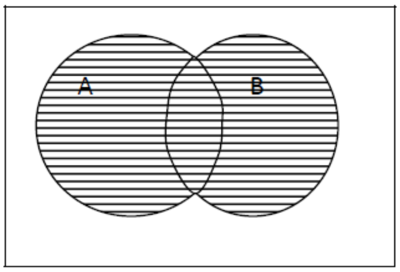
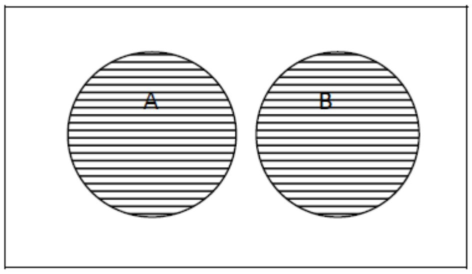
VENN DIAGRAM

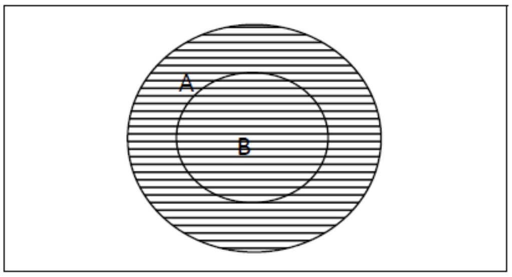
The relationship between sets can be represented by means of diagrams. It is known as Venn diagram. It consists of a rectangle and circles. Rectangle represents the universal set and circle represents any set.

For example A∪B, A∩B, A−B, and A^C can be represented as follows:

(1)A∪B

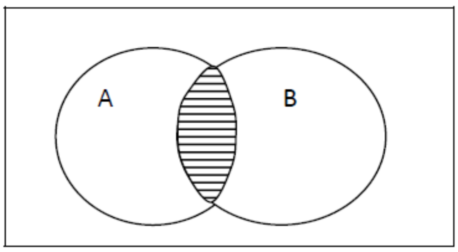


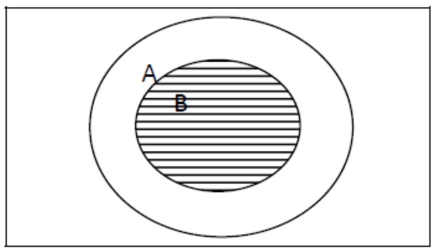


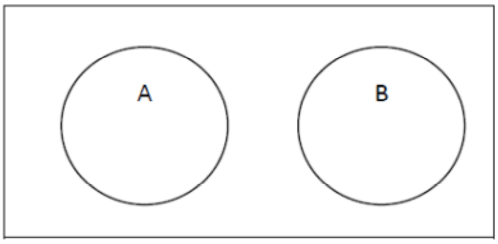


In diagram I A and B are intersecting in the second diagram, A and B are disjoint and in the third figure, B is a subset of A. In all the diagrams, A∪B is equal to the shaded area.

(ii) A∩B

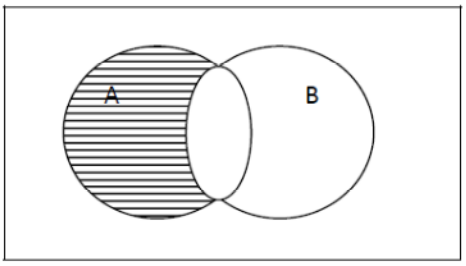






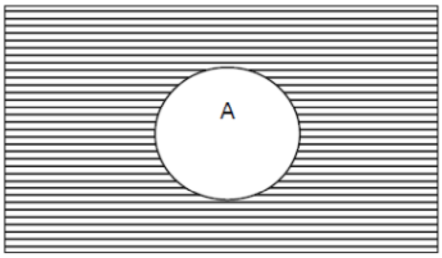
In first diagram A∩B is marked by lines. In the second diagram B is a subset of A and A∩B is also marked by lines. In the third diagram A and B are disjoint and therefore there is no intersection and so A∩B = ∅

(iii)A−B



A – B ie belongs to A but not in B is shaded by lines

(iv) A^c



A^c i.e. belongs to universe but not in A is shaded by lines

Theorems on Number of Elements in a Set

(i) n(A∪B) = n(A) + n(B) – n(A∩B)

(ii) n(A∪B∪C) = n(A) + n(B) + n(C) – n(A∩B) – n(A∩C) – n(B∩C) + n(A∩B∩C)

A∪B = at least one of them

A∩B = both A & B

A∪B∪C = At least one of them

A∩B∩C = All of them

1) Among 60 people, 35 can speak in English, 40 in Malayalam and 20 can speak in both the languages. Find the number of people who can speak at least one of the languages. How many cannot speak in any of these languages?

Answer

n(A) = Speak in English

n(B) = Speak in Malayalam

Given

$n(A) = 35,\ \ n(B) = 40$

$n(A\cap B) = 20$

$A\cup B$ = (ie people who speak in at least one of the language) =

$n(A) + n(B) - n(A\cap B)$

$= 35 + 40\ - 20 = 55$

Number of people who cannot speak in any one of these language = 60- 55 = 5

2) Each student in a class, studies at least one of the subject English, Mathematics and Accountancy. 16 study English, 22 Accountancy and 26 Mathematics. 5 study English and Accountancy, 14 study Mathematics and Accountancy and 2 English, Accountancy and Mathematics. Find the number of student who study

(i) English & Mathematics

(ii)English, Mathematics but not Accountancy

Answer

Let A = students study English

B = students study Mathematics

C = students study Accountancy

Given

$n(A\ ) = 16,\ n(B)\ 26,\ n(C)\ 22$

$n(A\cap C) = 5, n\ (B\cap C) = 14, n(A\cap B)?$

$n(A\cap B\cap C) = 2, n(A\cup B\cup C) = 40$

We know that

$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(A\cap C) - n(B\cap C) + n(A\cap B\cap C)$

$40 = 16 + 26 + 22 - n(A\cap B) - 5 - 14 + 12$

$n(A\cap B) = 16 + 26 + 22 - 5 - 14 + 2 - 40$

$= 7$

$\therefore$Number of students study for English & Mathematics = 7

Number of student who study English, Mathematics but not Accountancy = $n(A\cap B\cap C')$

$n(A\cap B\cap C') = n(A\cap B) - n(A\cap B\cap C)$

$= 7 - 2 = 5$

Number of student who study English, Mathematics and not Accountancy = 5
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3) In a college there are 20 teachers, who teach Accountancy or Statistics. Of these 12, teach Accountancy and 4 teach both Statistics and Accountancy. How many teach Statistics?

Answer

Let n(A) = teachers teach Accountancy

n(B) = teacher teach Statistics

Given n(A) = 12, n(B) ?

$n(A\cap B) = 4$, $n(A\cup B) = 20$

$n(A\cup B) = n(A) + n(B) - n(A\cap B)$

$20 = 12 + n(B) - 4$

$n(B) = 20 - 12 + 4 = 12$

Number of teachers teach Statistics = 12

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4) Out of 2400 students who appeared for BCom degree Examination, 1500 failed in Numerical skills, 1200 failed in Accountancy and 1200 failed in Informatics, 900 failed in both Numerical skills and Accountancy 800 failed in both Numerical skills and Informatics, 300 failed in Accountancy and Informatics, 40 failed in all subjects. How many students passed all three subjects?

Answer

Let A = number of students failed in Numerical Skills

B = number of students failed in Accountancy

C = number of students failed in Informatics

Given

$n(A) = 1500$, $n(B) = 1200$, $n(C) = 1200$

$n(A\cap B)$ 900, $n(A\cap C)$800, $n(B\cap C) = 300$

$n(A\cap B\cap C) = 40$

Number of students failed in at least one subject = $n(A\cup B\cup C)$

$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(A\cap C)- n(B\cap C) + n(A\cap B\cap C)$

$= 1500 + 1200 + 1200 - 900 - 800 - 300 + 40 = 1940$

Number of student passed in all subjects = 2400 – 1940 = 460

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Question

In a particular insurance life office, employees Smith, Jones, Williams and Brown have 'A' levels, with Smith and Brown also having a degree. Smith, Melville, Williams, Tyler, Moore and Knight are associate members of the Chartered Insurance Institute (ACII) with Tyler, and Moore having 'A' levels. Identifying set A as those employees with 'A' levels, set *C* as those employees who are ACII and set *D* as graduates:

a) Specify the elements of sets *A, C* and *D*.
b) Draw a Venn diagram representing sets A, C and D, together with their known elements.
c) What special relationship exists between sets A and D?
d) Specify the elements of the following sets and for each set, state in words what information is being conveyed.

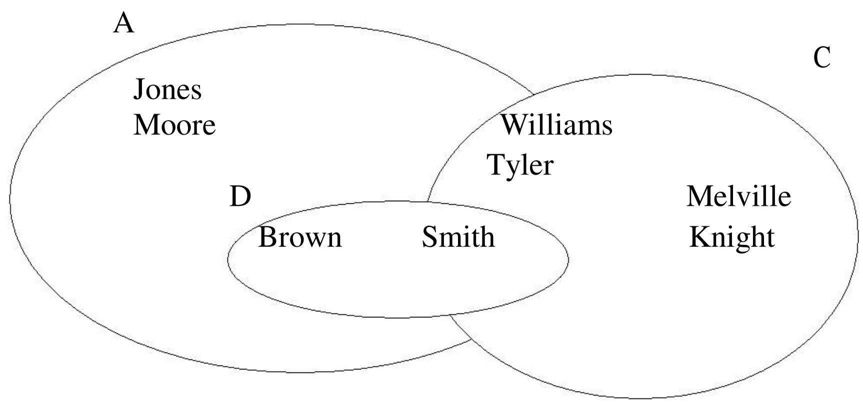
i. $A \cap C$ ii. $D \cup C$ iii. $D \cap C$

e) What would be a suitable universal set for this situation?

Answer

a) *A* = (Smith, Jones, Williams, Brown, Tyler, Moore);
C = (Smith, Melville, Williams, Tyler, Moore, Knight); D = (Smith, Brown)

b) The Venn diagram is shown in Figure 1.3.



c) From the diagram, it can be seen that *D* is a subset of *A*.

d) This information can be obtained either from the Venn diagram or from the sets listed in, a) above.

i. $A \cap C$ = (Williams, Tyler, Smith). This set gives the employees who have both '*A*' levels and are ACII.

ii. $D \cup C$ = (Brown, Smith, Williams, Tyler, Melville, Knight). This set gives the employees who are either graduates or ACII.

iii. $D \cap C$ = (Smith). This set gives the single employee who is both a graduate and ACII qualified.

e) A suitable universal set for this situation would be the set of all the employees working in the Life office.

Exercise:

1. Explain the different types of sets?
2. What is venn diagram?
3. If A is { 1,2,3,4} B={2,4,6,8} then find AUB, AnB,A-B
4. Verify Demorgan's Law for A= {2,3} B ={3,4} and U = {1,2,3,4,5}
5. Define Cartesion Products of two sets
6. Write the Relation 'a is the squ of b' in the set {1,3,5,9,10,25}
7. if A={a,b,c} and B= {x,y} then find A*B, B*A, A*A, B*B.
8. A town has a total population of 50,000 out of it 28,000 read hindu and 23,000 read manorama and the 4000 read both.indicate how many read neiher hindu nor manorama.

Let us Sum Up

This Lesson presented described about the set, set theory, Venn Diagrams, and its applications. A set is a collection of distinct objects, called elements, which are normally enclosed within brackets and separated by commas. Venn diagram is a pictorial representation of one or more sets. The Union and Intersection of sets were also discussed in detail. Some examples to understand the concept is also given in the Lesson.

MATRICES

Aims and Objectives

Matrices have applications in management disciplines like finance, production, marketing etc. Also in quantitative methods like linear programming, game theory, input-output models and in many statistical applications matrix algebra is used as the theoretical base. Matrix algebra can be used to solve simultaneous linear equations.

Matrices : Definition and Notations

A matrix is a rectangular array or ordered numbers. The term ordered implies that the position of each number is significant and must be determined carefully to represent the information contained in the problem. These numbers (also called elements of the matrix) are arranged in rows and columns of the rectangular array and enclosed by either square brackets, []; or parantheses ( ), or by pair of double vertical line || ||.

It is a rectangular presentation of numbers arranged systematically in rows and columns one number or functions are called the elements of the matrix. The horizontal lines of elements of the matrix are called rows and vertical lines of elements of matrix are called columns.

Order Of Matrix

A matrix having 'm' rows 'n' columns are called a matrix of order 'm x n' or simply 'm x n' matrix (read as an 'm' by 'n' matrix)

Types of Matrices

(i) **Rectangular matrix :** Any matrix with 'm' rows and 'n' column is called a rectangular matrix. It is a matrix of Order m x n. For example,

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 1 & 2 \end{bmatrix}$ is a 3 x 4 matrix

(ii) **Square matrix :** A matrix by which the number of rows are equal to the number of columns, is said to be a square matrix. Thus an m x n matrix is said to be square matrix if m= n and is known as a square matrix of order 'n'. For example,

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ is a square matrix of order 3

(iii) **Row matrix** : A matrix having only one row is called a row matrix. For example,

$A = \begin{bmatrix} 1 & 2 & 3 & 2 \end{bmatrix}$ is a row matrix.

(iv) **Column matrix** : A matrix having only column is called column matrix. For example,

$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ is a column matrix.

(v) **Diagonal matrix** : A square matrix is said to be diagonal it all elements except leading diagonal are zero. Elements a_{11}, a_{22}, a_{33} etc. termed as leading diagonal of a matrix. Example of Diagonal matrix is

$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 9 \\ 0 & 0 & 6 \end{bmatrix}$ is a diagonal matrix. Leading diagonal elements are 2, 3, 6.

(vi) **Scalar Matrix** : A diagonal matrix is said to be scalar matrix, if its diagonal elements are equal. For example.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(vii) **Unit matrix of identity matrix** : A diagonal matrix in which diagonal elements are 1 and rest are zero is called Unit Matrix or identity matrix. It is denoted by 1.

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a Unit matrix or Identity matrix.

(viii) **Null Matrix or Zero matrix:** A matrix is said to be zero or null matrix if all its elements are zero. For example

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a Null matrix or Zero matrix

(ix) **Triangular matrix:** If every element above or below the leading diagonal is zero, the matrix is called Triangular matrix. It may be upper triangular or lower triangular. In upper triangular all elements below the leading diagonal are zero and in the lower triangular all elements above the leading diagonal are zero. For example,

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is a matrix of upper triangular.

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ is matrix of lower triangular

(x) **Symmetric matrix :** Any square matrix is said to be symmetric if it is equal to transpose. That is, $A = A^t$

Transpose of a matrix as a matrix obtained by interchanging its rows and columns. It is denoted by A^t or A^1. Example of symmetric matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

(xi) **Skew Symmetric Matrix** : Any square matrix is said to be skew symmetric if it is equal to its negative transpose. That is $A = {}^{-}A^t$

For example $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} = A^t$

$$A^t = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$$

$$-A^t = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

Operation of matrices

Operation of matrices relate to the addition of matrices, difference, multiplication of matrix by a scalar and multiplication of matrices.

Addition of matrices : If A and B are any two matrices of the same order, their sum is obtained by the elements of A with the corresponding elements of B.
For example :

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -7 & 3 & 2 \\ -4 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 2 & 3 \\ -3 & -2 & 1 \\ 3 & -2 & 2 \end{bmatrix}$$

$$\text{Then } A + B = \begin{bmatrix} 3 & -4 & 5 \\ -10 & 1 & 3 \\ -1 & 1 & 4 \end{bmatrix}$$

Difference of Matrices : if A and B are, two matrices of the same order, then the difference is obtained by deducting the element of B from A.

$$\text{If} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$\text{Then } A - B = \begin{bmatrix} -2 & 3 & 0 \\ 3 & 3 & -1 \end{bmatrix}$$

Multiplication of a Matrix by a Scalar
The elements of Matrix A is multiplied by any value (ie. K) and matrix obtained is denoted by K

$$\text{For example : } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\text{Then } 5A = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 15 & 5 \\ 10 & 10 & 5 \end{bmatrix}$$

Practical Problems

1) If $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ - $B = \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ Find 3A – B ?

Ans: $3A = \begin{bmatrix} 0 & 6 & 9 \\ 6 & 3 & 12 \end{bmatrix}$

$$3A - B = \begin{bmatrix} 0 & 6 & 9 \\ 6 & 3 & 12 \end{bmatrix} - \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 & 6 \\ 5 & -1 & 7 \end{bmatrix}$$

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(2) Solve the equation:

$$2\begin{bmatrix} x & y \\ z & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 5\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

======

Ans : $2\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2z & 2t \end{bmatrix}$

$3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix}$

$5\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 \\ 20 & 30 \end{bmatrix}$

$\therefore \begin{bmatrix} 2x & 2y \\ 2z & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 \\ 20 & 30 \end{bmatrix}$

$2x + 3 = 15, 2x = 15 - 3 = 12, x = \frac{12}{2} = 6$

$2y + -3 = 25, 2y = 25 + 3 = 28,\ y = \frac{28}{2} = 14$

$2z + 0 = 20, 2z = 20,\ z = \frac{20}{2} = 10$

$2t + 6 = 30,\ 2t = 30 - 6 = 24,\ t = \frac{24}{2} = 12$

(3) Find the value of a, b if

$$2 \text{ x } \begin{bmatrix} a & 5 \\ 7 & b-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Ans : $2 \text{ x } \begin{bmatrix} a & 5 \\ 7 & b-3 \end{bmatrix} = \begin{bmatrix} 2a & 10 \\ 14 & 2b-6 \end{bmatrix}$

$\begin{bmatrix} 2a & 10 \\ 14 & 2b-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

$2a + 3 = 7,\ 2a = 7-3 = 4,\ a = \frac{4}{2} = 2$

$2b - 6 + 2 = 14, 2b = 14 + 6 - 2 = 18, b = \frac{18}{2} = 9$

Multiplication of two matrices

For multiplication, take each row of the left hand side matrix with all colums of the right hand side matrix. For example $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ Then $AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$

Multification of two Matrices :

If the number of columns in the first matrix is equal to the number of rows in the second matrix, then the matrices are compatible for multiplication. That is, if there are n columns in the first matrix then the number of rows in the second matrix must be n. Otherwise the matrices are said to be incompatible and their multiplication is not defined.

The matrices A,B are said to be comformable for multification when the number of columns of A = number of rows of B

When $A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and $B=\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$

then the product of AB is =

$$\begin{vmatrix} a_{11}b_{11+} a_{12}b_{21+} a_{13}ab_{31} & a_{11}b_{12+} a_{12}b_{22+} a_{13}ab_{32} & a_{11}b_{13+} a_{12}b_{23+} a_{13}ab_{33} \\ a_{21}b_{11+} a_{22}b_{21+} a_{23}ab_{31} & a_{21}b_{12+} a_{22}b_{22+} a_{23}ab_{32} & a_{21}b_{13+} a_{22}b_{23+} a_{23}ab_{33} \\ a_{31}b_{11+} a_{32}b_{21+} a_{33}ab_{31} & a_{31}b_{12+} a_{32}b_{22+} a_{33}ab_{32} & a_{31}b_{13+} a_{32}b_{23+} a_{33}ab_{33} \end{vmatrix}$$

Note: For multification ,take each row and multiply with all coloumns.

Practice:

Find the product of

1. $A=\begin{vmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 5 & 3 \end{vmatrix}$ and $B=\begin{vmatrix} 3 & 1 & 2 \\ 4 & 2 & 3 \\ 4 & -1 & 1 \end{vmatrix}$

   Find AB.

2. Two shops have the stock of large , medium and small sizes of a toothpaste.The number of each size stocked is given by the matrix A, where

$$A=\begin{array}{c} \text{Large} \quad \text{medium} \quad \text{small} \\ \begin{vmatrix} 150 & 240 & 120 \\ 90 & 300 & 210 \end{vmatrix} \end{array} \begin{array}{l} \\ \text{Shop no 1} \\ \text{Shop no 2} \end{array}$$

The cost matrix, B of the different size of the tooth paste is given by

$$B=\begin{array}{c} \text{Cost Rs} \\ \begin{vmatrix} 14 \\ 10 \\ 6 \end{vmatrix} \end{array} \begin{array}{l} \\ \text{large} \\ \text{medium} \\ \text{small} \end{array}$$

Find the investment in tooth paste by each shop

3.Three shop keepers A,B and C go to a store to buy stationery. A purchases 12 dozen note book, 5 dozen pens and 6 dozen pencils. B purchases 10 dozen note books, 6 dozen pens and 7 dozen pencils. C purchases 11 dozen note book, 13 dozen pens, and 8 dozen pencils. A note book costs 40 paise, a pen costs ₹1.25 and pencil costs 35 paise. Use matrix multiplication to calculate each individual bill ?

Ans : Bill of purchase = Purchase Quantity x Price

Practical Problems

(1) Let A = $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix}$ B= $\begin{bmatrix} 2 & 3 & 1 \\ 5 & 4 & 2 \\ 1 & 5 & 3 \end{bmatrix}$ Compute AB

Ans: AB = $\begin{bmatrix} 1x2 + 2x5 + 3x1 & 1x3 + 2x4 + 3x5 & 1x1 + 2x2 + 3x3 \\ -2x2 + 1x5 + 4x1 & -2x3 + 1x4 + 4x5 & -2x1 + 1x2 + 4x3 \end{bmatrix}$

AB = $\begin{bmatrix} 2 + 10 + 3 & 3 + 8 + 15 & 1 + 4 + 9 \\ -4 + 5 + 4 & -6 + 4 + 20 & -2 + 2 + 12 \end{bmatrix}$

AB = $\begin{bmatrix} 15 & 26 & 14 \\ 5 & 18 & 12 \end{bmatrix}$

(2) Let A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$ Find AB and BA and hence show that AB≠ BA

Ans : AB = $\begin{bmatrix} 1x - 2 + 2x1 & 1x2 + 2x - 1 \\ 3x - 2 + 4x1 & 3x2 + 4x - 1 \end{bmatrix}$

$\begin{bmatrix} -2 + 2 & 2 + -2 \\ -6 + 4 & 6 + -4 \end{bmatrix}$ = $\begin{bmatrix} 0 & 0 \\ -2 & 2 \end{bmatrix}$

BA = $\begin{bmatrix} -2x1 + 2x3 & -2x2 + 2x4 \\ 1x1 + -1x3 & 1x2 + -1x4 \end{bmatrix}$

= $\begin{bmatrix} -2 + 6 & -4 + 8 \\ 1 + -3 & 2 + -4 \end{bmatrix}$ = $\begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix}$

Therefore, <u>AB≠ BA</u>

(3) Let A= $\begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

Calculate AC, BC and (A+B)C and verify that (A+B)C = AC + BC.

Ans: AC = $\begin{bmatrix} 0 & - & 12 & + & 21 \\ -12 & + & 0 & + & 24 \\ 14 & + & 16 & + & 0 \end{bmatrix}$ = $\begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix}$

BC = $\begin{bmatrix} 0 & - & 2 & + & 3 \\ 2 & + & 0 & + & 6 \\ 2 & - & 4 & + & 0 \end{bmatrix}$ = $\begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix}$

A+ B = $\begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix}$

$$(A+B)\,C = \begin{bmatrix} 0 & - & 14 & + & 24 \\ -10 & + & 0 & + & 30 \\ 16 & + & 12 & + & 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$$

$$AC + BC = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$$

$\therefore$ $\underline{(A+B)\,C = AC + BC}$

(4) Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ Show that $\underline{A^3 - 23A - 40\,I = 0}$

Ans : $A^3 = A x\, A x\, A$

$A^2 = A x\, A$

$A^3 = A^2 \text{ x } A$

$$A^2 = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

$$23A = \begin{bmatrix} 23 & 46 & 69 \\ 69 & -46 & 23 \\ 92 & 46 & 23 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$40\,I = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$A^3 - 23A - 40\,I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{\therefore\ A^3 - 23A - 40\,I = 0}$

(5) Let A = $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ B = $\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$ What is the value of 'k' if any make AB = BA

Ans: AB = $\begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix}$

$$BA = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$$

AB = BA

$-10 + 5k = 15$

$5k = 15 + 10 = 25$

$\therefore k = \frac{25}{5} \quad \underline{\underline{= \ 5}}$

(6) Two shops have the stock of large, medium and small size of a tooth paste. The number of each size stocked is given by the matrix A, where

$$A = \begin{array}{c} \\ \end{array}\begin{bmatrix} large & medium & small \\ 150 & 240 & 120 \\ 90 & 300 & 210 \end{bmatrix} \begin{array}{l} \\ Shop\ No.1 \\ Shop\ No.2 \end{array}$$

are cost matrix 1 of the different size of the tooth paste is given by cost (₹)

$$B = \begin{bmatrix} 14 \\ 10 \\ 6 \end{bmatrix} \begin{array}{c} Large \\ medium \\ small \end{array}$$

Find the investment in the toothpaste by each shop.

Ans : Investment = AB

$$AB = \begin{bmatrix} 150 & 240 & 120 \\ 90 & 300 & 210 \end{bmatrix} X \begin{bmatrix} 14 \\ 10 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2100 & + & 2400 & + & 720 \\ 1260 & + & 3000 & + & 1260 \end{bmatrix} = \begin{bmatrix} 5220 \\ 5520 \end{bmatrix}$$

Investment in toothpaste by

Shop 1 = $\underline{\underline{5220}}$

Shop 2 = $\underline{\underline{5520}}$

(7) In a large legislative Assembly electron, a political group hired a public relations firm to promote its candidate in three ways; telephonic, housecalls, and letters. The cost per contract (in paise) is given in matrix A as.

$$A = \begin{array}{c} \textit{Cost per Contract} \\ \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{array} \begin{array}{c} \\ \textit{Telephone} \\ \textit{House call} \\ \textit{Letter} \end{array}$$

The number of contract of each type made in two cities X and Y is given by

$$B = \begin{bmatrix} \textit{Telephone} & \textit{House calls} & \textit{Letter} \\ 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{array}{c} \\ x \\ y \end{array}$$

Find the total amount spent by the group in the two cities x and y ?

Amount spent = BA

$$BA = \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} X \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix}$$

$$\begin{bmatrix} 40000 + 50000 + 2,50,000 \\ 1,20,000 + 1,00,000 + 5,00,000 \end{bmatrix}$$

$$= \begin{bmatrix} 3,40,000 \\ 7,20,000 \end{bmatrix}$$

Amount spent by

City X = 3,40,000 paise i.e. ₹3400/-

City Y = 7,20,000 paise i.e. ₹7200/-

(8) Three shop keepers A,B and C go to a store to buy stationery. A purchases 12 dozen note book, 5 dozen pens and 6 dozen pencils. B purchases 10 dozen note books, 6 dozen pens and 7 dozen pencils. C purchases 11 dozen note book, 13 dozen pens, and 8 dozen pencils. A note book costs 40 paise, a pen costs ₹1.25 and pencil costs 35 paise. Use matrix multiplication to calculate each individual bill ?

Ans : Bill of purchase = Purchase Quantity x Price

Let A = Purchase Quantity

B = Price

$$\text{Then } A = \begin{bmatrix} \textit{Note book} & \textit{Pens} & \textit{pencil} \\ 12 \times 12 & 5\text{x}12 & 6\text{x}12 \\ 10 \times 12 & 6\text{x}12 & 7\text{x}12 \\ 11\text{x}12 & 13\text{x}12 & 8\text{x}12 \end{bmatrix} \begin{array}{c} \\ \textit{Purchase A} \\ \textit{B} \\ \textit{C} \end{array}$$

$$\text{Then } A = \begin{bmatrix} 144 & 60 & 72 \\ 120 & 72 & 84 \\ 132 & 156 & 96 \end{bmatrix}$$

$$B = \begin{bmatrix} 40 \\ 125 \\ 35 \end{bmatrix} \quad ₹1.25 = 125\ paise$$

$$AB = \begin{bmatrix} 144x40 & + & 60 \text{ x } 120 & + & 72 \text{ x } 35 \\ 120x\ 40 & + & 72 \text{ x } 125 & + & 84 \text{ x } 35 \\ 132 \text{ x } 40 & + & 156 \text{ x } 125 & + & 96 \text{ x } 35 \end{bmatrix}$$

$$AB = \begin{bmatrix} 15780 \\ 16740 \\ 28140 \end{bmatrix}$$

Bill of the Shop keeper A = <u>15780/-</u>

B = <u>16740/-</u>

C = <u>28140/-</u>

Determinants

A determinant is a compact form showing a set of numbers arranged in rows and columns, the number of rows and the number of columns being equal. The number in a determinant are known as the elements of the determinant.

Matrics which are not square do not have determinants.

Determinant of Square matrix of order 1

The determinants of 1 x 1 matrix A [a] is denoted by |A| or det. A (i.e. determinant of A) and its value is a.

Determinant of Square matrix of order 2

Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix of order 2 x 2

Then the determinant A is defined as

|A| = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = ad – bc

Determinant with 3 rows and columns

Let A = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a matrix of order 3 x 3.

Then the determinant A is defined as

$|A| = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} \; c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$ i.e.

a(ei-uf) - b(di-gf) c (dh-ge)

Practical Problems

1. Evaluate the determinant

$\begin{vmatrix} 2 & -3 \\ 4 & 9 \end{vmatrix}$

Ans : $\begin{vmatrix} 2 & -3 \\ 4 & 9 \end{vmatrix} = 2 \times 9 - 4 \times -3$

$= 18 + 12 \; \underline{\underline{= \; 30}}$

2. Find the value of the determinant

$\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$

Ans : $\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = 1 \begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} - 3 \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$

$1\,(-4 - 4\,) - 2\,(\,8 - 6) - 3\,(4 - -3)$

$1\,(-8) - 2\,(2) - 3\,(7)$

$= -8 - 4 - 21 \; \underline{\underline{= -33}}$

Singular and Non singular matrices – A square matrix 'A' is said to be singular if its determinant value is zero. If $|A| \neq 0$, then A is called non-singular.

Minor elements of a matrix:

Minor element is the determinant obtained by deleting its rows and the column in which element lies.

Example – (1) Find the Minor of element 6 in the determinant A = $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Ans : Minor of 6 = $\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 1 \times 8 - 2 \times 7$

$= 8 - 14 \; \underline{\underline{= -6}}$

2) If A = $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \\ 4 & 2 & 2 \end{vmatrix}$ Find the minor of 3

Answer : Minor of 3 = $\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}$ = 1 x 2 – 0 x 2 = 2 – 0 = 2

Co-factor of an element

Co-factor of an element is obtained by multiplying the minor of that element with $(-1)^{(i+j)}$. where i = the row in which the element belongs, s = the column in which the element belongs.

Co-factpr of an element = Minor of an element X $(-1)^{i+j}$

Example 1. Find the Co-factors of all the element of the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Ans : Minor element

1 = 3, -2 = 4

4 = -2, 3= 1

Co-factors 1 = 3 x -1^{1+1} = 3 x -1^2 = 3

-2 = 4 x -1^{1+2} = 4 x -1^3 = -4

4 = -2 x -1^{2+1} = -2 x -1^3 = 2

3 = 1 x -1^{2+2} = 1x -1^4 = 1

2) Find the co-factors of the elements of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify that

$a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 0$

Ans : Minor of an element :

2 = $\begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix}$ = (0 x -7) – (4 x 5) = 0 – 20 = -20

-3 = $\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix}$ = (6 x -7) – (4 x 1) = -42 – 4 = -46

5 = $\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix}$ = (6 x 5) – (0 x 15) = 30 – 0 = 30

6 = $\begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix}$ = (-3 x -7) – (5 x 5) = 21 – 25 = -4

0 = $\begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix}$ = (2 x -7) – (5 x 1) = -14 – 5 = -19

4 = $\begin{vmatrix} 2 & -3 \\ 1 & -5 \end{vmatrix}$ = (2 x 5) – (-3 x 1) =10 – -3 = 13

1 = $\begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix}$ = (-3 x 4) – (5 x 0) = -12 – 0 = -12

$5 = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = (2 \times 4) - (5 \times 6) = 8 - 30 = -22$

$-7 = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = (2 \times 0) - (-3 \times 6) = 0 - {}^{-}18 = 18$

Co-factors:

$2 = -20 \times -1^{1+1} = -20 \times -1^{2} = \underline{\underline{-20}}$

$-3 = -46 \times -1^{1+2} = -46 \times -1^{3} = \underline{\underline{46}}$

$5 = 30 \times -1^{1+3} = 30 \times -1^{4} = \underline{\underline{30}}$

$6 = -4 \times -1^{2+1} = -4 \times -1^{3} = \underline{\underline{4}}$

$0 = -19 \times -1^{2+2} = -19 \times -1^{4} = \underline{\underline{-19}}$

$4 = 13 \times -1^{2+3} = 13 \times -1^{5} = \underline{\underline{-13}}$

$1 = -12 \times -1^{3+1} = -12 \times -1^{4} = \underline{\underline{-12}}$

$5 = -22 \times -1^{3+2} = -22 \times -1^{5} = \underline{\underline{22}}$

$-7 = 18 \times -1^{3+3} = 18 \times -1^{6} = \underline{\underline{18}}$

$a_{11} = 2, \quad a_{12} = -3, \quad a_{13} = 5$

$A_{31} = -12, \quad A_{32} = 22, \quad A_{33} = 18$

$a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 0$

i.e., $= 2 \times -12 + -3 \times 22 + 5 \times 18$

$= -24 + -66 + 90$

$= \underline{\underline{-90 + 90 = 0}}$

Adjoint Matrix

Adjoint of a given matrix is the transpose of the matrix formed by co-factors of the elements. It is denoted by Adj A.

Let A $= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then Adj A = Transpose $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Practical Problems

1) Find adj A for A = $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Ans: Minor element:

2 = 4, 3 = 1, 1 = 3, 4 = 2

Co-factors:

$2 = 4 \times -1^{1+1} = 4,\quad 3 = 1 \times -1^{1+2} = -1$

$1 = 3 \times -1^{2+1} = -3,\quad 4 = 2 \times -1^{2+2} = 2$

$$\text{adj A} = \text{Transpose} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

2) Find adj A for A = $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Ans: Minor element:

2 = -1, 1 = 7, 3 = 5

3 = -3, 1 = 3, 2 = 3

1 = -1, 2 = -5, 3 = -1

Co-factor elements

$2 = -1 \times -1^{1+1} = -1,\quad 1 = 7 \times -1^{1+2} = -7$

$3 = 5 \times -1^{1+3} = 5$

$3 = -3 \times -1^{2+1} = 3,\quad 1 = 3 \times -1^{2+2} = 3$

$2 = 3 \times -1^{2+3} = -3$

$1 = -1 \times -1^{3+1} = -1,\quad 2 = -5 \times -1^{3+2} = 5$

$3 = -1 \times -1^{3+3} = -1$

$$\text{adj A} = \text{Transpose} \begin{bmatrix} -1 & -7 & 5 \\ 3 & 3 & -3 \\ -1 & 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

Invertible Matrix and Inverse of a Matrix

Let A be a square matrix of order n, if there exist a square matrix B of order n, such that AB = BA = I

Then A is said to be convertible and B is called on inverse of A and A is called inverse of B

Where I = Identity Matrix

Inverse of A is denoted by A^{-1}

$$A^{-1} = \frac{1}{|A|} \text{adj } A \text{ or}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

1) Find the inverse matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Ans: $|A| = (2 \times 3 - 1 \times -1) = 6 - -1 = 7$

Minor element:

$2 = 3, \quad -1 = 1, \quad 1 = -1, \quad 3 = 2$

Co-factors element

$2 = 3 \times -1^{1+1} = 3, \quad -1 = 1 \times -1^{1+2} = -1$

$I = {}^{-}1 \times {}^{-}1^{2+1} = 1, \quad 3 = 2 \times {}^{-}1^{2+2} = 2$

$$\text{adj } A = \text{Transpose} \begin{bmatrix} 3 & {}^{-}1 \\ 1 & 2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{7} \begin{bmatrix} 3 & {}^{-}1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{7} & \frac{{}^{-}1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

2. Compute the inverse of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Ans. $|A| = 1(3\text{-}1) - 2(2 - {}^{-}1) + 5(2 - {}^{-}3)$

$= 1(2) - 2(3) + 5(5)$

$= 2 - 6 + 25 = 21$

Minor element:

$1 = 2, 2 = 3, 5 = 5$

$2 = {}^{-}3, 3 = 6, 1 = 3$

${}^{-}1 = {}^{-}13, 1 = {}^{-}9, 1 = {}^{-}1$

Co-factors element

$1 = 2 \times {}^{-}1^{1+1} = 2, \quad 2 = 3 \times {}^{-}1^{1+2} = {}^{-}3, \quad 5 = 5 \times {}^{-}1^{1+3} = 5$

$2 = {}^{-}3 \times {}^{-}1^{1+2} = 3, 3 = 6 \times {}^{-}1^{2+2} = 6, 1 = 3 \times {}^{-}1^{2+3} = {}^{-}3$

$-1 = -13 \times -1^{3+1} = -13, \quad 1 = -9 \times -1^{3+2} = 9, \quad 1 = -1 \times -1^{3+3} = -1$

$$\text{Adj A} = \text{Transpose} \begin{bmatrix} 2 & -3 & 5 \\ 3 & 6 & -3 \\ -13 & 9 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj A})$$

$$= \frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{21} & \frac{3}{21} & \frac{-13}{21} \\ \frac{-3}{21} & \frac{6}{21} & \frac{9}{21} \\ \frac{5}{21} & \frac{-3}{21} & \frac{-1}{21} \end{bmatrix}$$

3) If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then verify that A adjA = $|A|1$. Also find A^{-1}

Ans: $|A| = 1(16 - 9) - 3(4 - 3) + 3(3 - 4)$

$= 1(7) - 3(1) + 3(-1)$

$= 7 - 3 + -3 = 1$

$$\text{adj } A = \text{Transpose} \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A(\text{adj } A) = |A|1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A\,(\text{adj } A = |A|1$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

4) If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$

Ans: $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

$|AB| = (14 - 25) = -11$

$(AB)^{-1} = \frac{1}{|AB|}(\text{adj}\,(AB))$

adj (AB) = adj A x adj B

adj A:

Minor element:

2 = -4, 3 = 1, 1 = 3, -4 = 2

Co-factors element

2 = -4, 3 = -1, 1 = -3, -4 = 2

$\text{adj}\,A = \text{Transpose}\begin{bmatrix} -4 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$

$\text{adj}\,B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{-11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

$|A| = -11, \ |B| = 1$

$A^{-1} = \frac{1}{-11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$

$B^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$B^1A^1 = \frac{1}{-11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$

$= = \frac{1}{-11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

Hence $(AB)^{-1} = B^1A^1$

Solving simultaneous equations with the help of Matrices

Firstly, express the equation in the form of AX = B

Then possibilities

When $|A| \neq 0$

Then $X = A^{-1}B$ i.e., the system has a unique solution.

∴ the system is consistant

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

When $|A| = 0$

Then we calculate (adj A)B

If (adj A)B = 0, then the system will have infinite solution were the system is consistent.

If (adj A)B ≠ 0, then the system will have no solution.

Problem

1) Solve the linear equation by using matrix

$$5x + 2y = 4$$

$$7x + 3y = 5$$

Ans: $AX = B$

Let $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$|A| = (15 - 14) = 1$

i.e., $1 \neq 0$

Then $X = A^{-1}B$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

<u>Adj A:</u>

Minor element 5 = 3, 2 = 7, 7 = 2, 3 = 5

Co-factors element 5 = 3, 2 = -7, 7 = -2, 3 = 5

adj A = Transpose $\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$X = \begin{bmatrix} 12 & -10 \\ -28 & 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$x = \underline{\underline{2}}$ $\quad$ $y = \underline{\underline{-3}}$

2) Solve the equation by using matrix

$x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

Ans: $AX = B$

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

$|A| = 1(1+3) - (-1)(2+3) + 1(2-1)$

$= 1(4) + 1(5) + 1(1)$

$= 4 + 5 + 1 = 10$ $\quad$ ie $\neq 0$

Then $X = A^{-1}B$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

Factor elements:

1 = 4,	-1 = -5,	1 = 1
2 = 2,	1 = 0	-3 = -2
1 = 2,	1 = 5,	1 =3

$$\text{Adj A} = \text{Transpose}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$X = A^{-1}B$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 16 + & 0 + & 4 \\ -20 + & 0 + & 10 \\ 4 + & 0 + & 6 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

i.e., $\underline{\underline{x = 2,\ y = -1,\ z = 1}}$

3) Solve the following equation by using matrix

$5x - 6y + 4z = 15$

$7x + 4y - 32 = 19$

$2x + y + 6z = 46$

Ans: $AX = B$

Let $A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$B = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$

$|A| = 5(24 - {}^{-}3) - {}^{-}6\ (42 - {}^{-}6) + 4\ (7 - 8)$

$= 5(27) + 6(48) + 4(-1)$

$= 135 + 288 - 4 = 419$

Then $X = A^{-1}B$

$$A^{-1} = \frac{1}{|A|}\ (\text{adj } A)$$

Co-factor elements :

5 = 27, -6 = -48, 4 = -1

7 = 40, 4 = 22, -3 = -17

2 = 2, 1 = 43, 6 = 62

$$\text{Adj } A = \text{Transpose} \begin{bmatrix} -27 & -48 & -1 \\ 48 & 22 & -17 \\ 2 & 43 & 62 \end{bmatrix} = \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

$X = A^{-1}B$

$$X = \frac{1}{|A|}\ (\text{adj } A)\ \ B$$

$$= \frac{1}{419} \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix} \times \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$$= \frac{1}{419} \begin{bmatrix} 1257 \\ 1676 \\ 2514 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$X = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}$$

$\therefore\ \underline{\underline{x = 3, y = 4,\ z = 6}}$

Properties of Matrix Addition

If A,B and C are three matrices of same dimension, then,

1. matrix addition is commutative, i.e. $A + B = B + A$
2. matrix addition is associative, i.e, $(A+B)+C = A+(B+C)$
3. zero matrix is the additive identity, i.e, $A+\mathbf{0} = A$
4. B is an additive inverse if $A+B = \mathbf{0}$.

Properties of multiplication

1. Matrix multiplication, in general, is not commutative. i.e, $AB \neq BA$.
2. Matrix multiplication is associative. i.e., $A(BC) =(AB)C$
3. Matrix multiplication is distributive, i.e, $A(B+C) = AB + AC$

Properties of Transpose

1. Transpose of a sum (or difference) of two matrices is the sum (or difference) of the transposes, i.e. $(A \pm B)^T = A^T \pm B^T$
2. Transpose of transpose is the original matrix. i.e. $(A^T)^T = A$
3. The transpose of a product of two matrices is the product of their transposes taken in reverse order. i.e., $(AB)^T = B^T A^T$

Properties of determinants

Following are the useful properties of determinants of any order. These properties are very useful in expanding the determinants.

1. The value of a determinant remains unchanged. If rows are changed into column and columns into rows, i.e.
   $|A| = |A^t|$
2. If two rows (or columns) of a determinant are interchanged, then the value of the determinant so obtained is the negative of the original determinant.
3. If each element in any row or column of a determinant is multiplied by a constant number say K, then the determinant so obtained is K times the original determinant.
4. The value of a determinant in which two rows (or columns) are equal is zero.
5. If any row (or column) of a determinant is replaced by the sum of the row and a linear combination of other rows (or columns), then the value of the determinant so obtained is equal to the value of the original determinant.
6. The rows (or columns) of a determinant are said to be linearly dependent if $|A|=0$, otherwise independent.

Let us Sum Up

Matrices play an important role in quantitative analysis of managerial decision. They also provide very convenient and compact methods of writing a system of linear simultaneous equations and methods of solving them. These tools have also become very useful in all functional areas of management. Another distinct advantage of matrices is that once the system of equations can be set up in matrix form, they can be solved quickly using a computer. A number of basic matrix operations (such as matrix addition, subtraction, multiplication) were discussed in this Lesson.